Model Paper

GOVERNMENT COLLEGE UNIVERSITY, FAISALABAD

BA/BSc (Part-I)  Annual 2017
Subject: Mathematics (A-Course)  Paper: (I)  Course Code: MTH-301
Course Title: Calculus and Analytic Geometry
Time Allowed: 03:00 Hours  Maximum Marks: 100  Pass Marks: 33%

Note: Attempt Six questions in all, selecting two question from section I and II each and one question from III and IV.

Section-I

Question # 1: (a) Show that \( \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \). \(9,8\)

(b) If \( p_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \), then prove that \( \lim_{x \to a} p_n(x) = p_n(a) \).

Question # 2: (a) Let \( v \) be the velocity of a particle at any given time \( t \). Deduce that the acceleration of the particle at this instant is \( \frac{dv}{dt} \). \(9,8\)

(b) Find the root of the equation \( 4 \sin x = e^x \) in the interval \((0, 0.5)\).

Question # 3: (a) Show that \( \frac{d^n}{dx^n} \left( \frac{n!}{x^n} \right) = \frac{(-1)^n n!}{x^{n+1}} \left[ \ln x - \frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n} \right] \). \(9,8\)

(b) Use L’Hospital rule to prove that \( \lim_{x \to a} \left[ \frac{e^x + b^x}{2} \right] = \sqrt{ab}, a > 0, b > 0 \).

Question # 4: (a) Apply Taylor’s theorem to prove that \( (a + b)^m = a^m + \frac{m!}{1!} a^{m-1} b + \frac{m(m-1)!}{2!} a^{m-2} b^2 + \ldots \) for all real \( m, a > 0, -a < b < a \).

(b) Evaluate the given limits: \( \lim_{x \to 0} \frac{\tanh x - \sinh x}{x^3} \). \(9,8\)

Section-II

Question # 5: (a) Find equations of tangent and normal to each of the following curves at the indicated point: \( c^2(x^2 + y^2) = x^2y^2 \) at \( \left( \frac{c}{\cos \theta}, \frac{c}{\sin \theta} \right) \).

(b) Find the pedal equation of \( r^n = a^n \cos n\theta \). \(9,8\)

Question # 6: (a) Show that \( \tan y = \frac{dy}{dx} \cdot \frac{x}{y} + x \). \(9,8\)

(b) Sketch the graph of the given curves \( r^2 = a \sin 2\theta \).

Question # 7: (a) Show that the shortest between the lines \( x + a = 2y = -12z \) and \( x = y + 2a = 6(z - a) \) is \( 2a \). \(9,8\)

(b) Express the given equation in cylindrical coordinates: \((x - 7)^2 - z^2 + 4 = 0\). \(9,8\)
Question # 8: (a) Find an equation of the cone whose directrix is
   \[ y^2 = x, \ z = 4 \] and whose vertex is at \( (0,2,0) \)
   (b) Find an equation of the tangent plane to the sphere \( x^2 + y^2 + z^2 - 4x + 2y - 6z = 0 \)
   at the point \( P(3,2,5) \)

Section III

Question # 9: (a) Find the asymptotes of \( 2xy + 2y = (x-2)^2 \)
   (b) Show that the height of an open cylinder of given surface \( S \) and greatest
   Volume is equal to the radius of the base.

Question # 10: (a) Find the equations of the tangents at the multiple points of the curve
   \( (y-z)^2 = x(x-1)^2 \)
   (b) Find the intrinsic equation of \( \gamma = a(1-\cos \theta) \)

Section IV

Question # 11: (a) Evaluate \( \int x^5 e^{x^3} \, dx \).
   (b) Integrate \( \frac{1}{x^4 + 1} \) with respect to \( x \).

Question # 12: (a) Calculate \( \int_0^{\pi/2} \ln \sin x \, dx \).
   (b) Use Trapezoidal Rule to approximate the integral \( \int_0^\pi \sin x \, dx \) with \( n = 6 \).
Note: Attempt Six questions in all, selecting two questions from section I & IV each and one question from II & III.

SECTION-I

Q. 1 (a) Prove that the product of matrices.

\[ A = \begin{bmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & \sin^2 \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos^2 \beta & \cos \beta \sin \beta \\ \cos \beta \sin \beta & \sin^2 \beta \end{bmatrix} \]

is the zero matrix if \( \alpha \) and \( \beta \) differ by \( \frac{(2n+1)\pi}{2} \), where \( n \) is an integer.

(b) Prove that

\[ \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2 + 2a & 2a + 1 & 1 \\ a & 2a + 1 & a + 2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6. \]

Q. 2 (a) A yarn merchant sells brands A, B, C of yarn each of which is a blend of Pakistani, Egyptian and American cotton in ratio 1:2:1, 2:1:1 and 2:0:2, respectively. If one kilogram of A, B, C costs 40,50 and 60 rupees respectively, find the cost of a kilogram of cotton of each country.

(b) Let A and B be nonsingular matrices of the same order then AB is a nonsingular. Then prove that \((AB)^{-1} = B^{-1}A^{-1}\).

Q. 3 (a) Show that the vectors \( \{(1,2,3),(0,1,2),(0,0,1)\} \) generate \( \mathbb{R}^3 \).

(b) Let \( U \) and \( W \) be subspaces of a vector space \( V \) over a field \( F \). Then prove that \( U \cap W \) is also a subspace of \( V \).

Q. 4 (a) A linear transformation \( T : U \rightarrow V \) is one-one if and only if \( \text{N(T)} = \{0\} \).

(b) Find a basis and dimension of \( \text{R(T)} \) where \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is defined by

\[ T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, x_2, x_1 + x_2 - 2x_3). \]

SECTION-II

Q. 5 (a) Let \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) belong to \( \mathbb{R}^2 \) verify that

\[ \langle u, v \rangle = u_1v_1 - 2u_1v_2 - 2u_2v_1 + 5u_2v_2 \] is an inner product on \( \mathbb{R}^2 \).

(b) Show that the rows (columns) of the matrix

\[ \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

form an orthonormal set.
Q.6  (a) Find the eigen values and corresponding eigen vectors of the matrix \[
\begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & -1 \\
-1 & 1 & 4
\end{pmatrix}
\] (8,8)

(b) If \( \lambda \) is an eigen value of an orthonormal matrix, then \( |\lambda| = 1 \).

SECTION-III

Q.7  (a) Solve \( x \frac{dy}{dx} + y = y^2 \ln x \). (8,8)

(b) Solve \( p = \tan \left( x - \frac{p}{1 + p^2} \right) \).

Q.8  (a) Solve \( (xy^2 + 2x^2 y^3) dx + (x^2 y - x^3 y^2) dy = 0 \). (8,8)

(b) Find an equation of orthonormal trajectory of the curve \( r^n \cos n \theta = a^n \).

SECTION-IV

Q.9  (a) Solve \( (D^3 - D^2 + D - 1)y = 4 \sin x \). (9,8)

(b) Solve \( x^3 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^3 \frac{dy}{dx} + xy = 1 \).

Q.10 (a) Solve particular solution: \( \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = \left( 1 + e^{-x} \right)^2 \). (9,8)

(b) Solve: \( \frac{d^3 y}{dx^3} + y = \sec^3 x \).

Q.11 (a) Compute (i) \( L(\sin at) \) (ii) \( L^{-1} \left( \frac{3s + 1}{s^2 - 6s + 18} \right) \). (9,8)

(b) Apply the power series method to solve: \( y'' - ky = 0 \).

Q.12 Use Laplace transformation method to solve:

(a) \( \frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + y = te^t; \quad y(0) = 0 = y'(0) \).

(b) \( \frac{dy}{dt} + 4y = 2e^t - 4e^{-t}; \quad y(0) = 0 \).